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ScienceDirect
International Journal of Heat and Mass Transfer 50 (2007) 26–35
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Forced convection with viscous dissipation in the thermal entrance region of a circular duct with prescribed wall heat flux

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Received 7 November 2005
Available online 7 September 2006

Abstract

The thermal entrance forced convection in a circular duct with a prescribed wall heat flux distribution is studied under the assumptions of a fully developed laminar flow and of a negligible axial heat conduction in the fluid, by taking into account the effect of viscous dissipation. The solution of the local energy balance equation is obtained analytically by employing the Laplace transform method. The effect of viscous dissipation is taken into account also in the region upstream of the entrance cross-section, by assuming an adiabatic preparation of the fluid. The latter hypothesis implies that the initial condition in the entrance cross-section is a non-uniform radial temperature distribution. Two special cases are investigated in detail: an axially uniform wall heat flux, a wall heat flux varying linearly in the axial direction.

Keywords: Forced convection; Duct flow; Viscous dissipation; Analytical methods; Laplace transform

1. Introduction

The effect of viscous dissipation may become very important in several flow configurations occurring in the engineering practice. In fact, viscous dissipation affects strongly the heat transfer process whenever the operating fluid has a low thermal conductivity, a high viscosity and flows in ducts with a small cross-section and a small wall heat flux. All these features may occur, for instance, in the microchannel flows considered for the design of MEMS. As is well known, the effect of viscous heating increases with the square of the mass flow rate and, as a consequence, becomes specially important under conditions of forced convection.

A traditional arena for predictions of the viscous dissipation effect in duct flows is the analysis of the laminar thermal entrance regime. Several duct geometries have been investigated, even if most of the published papers refer to the circular duct or to the parallel-plate channel. The thermal entrance problem with viscous dissipation has been investigated by Brinkman [1] with reference to uniform wall temperature or adiabatic wall boundary conditions. Further analyses have been performed by Ou and Cheng [2–4], Lin et al. [5] and Basu and Roy [6]. The latter papers include the study of the boundary conditions of uniform wall heat flux [2,6] and of external convection (third kind boundary condition) [5]. The solutions presented by these authors are extensions of the classical Graetz–Nusselt solution, obtained in the absence of internal heat source terms and widely treated in the literature [7]. The main consequence of the viscous dissipation effect is in the evaluation of the local Nusselt number. Indeed, it has been pointed out that this quantity may become singular at some

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doi:10.1016/j.ijheatmasstransfer.2006.06.036

Nomenclature

a, b	arbitrary complex numbers, Eq. (35)
Br	$\mu u_m^2/(r_0 q_{w0})$, Brinkman number
c_p	specific heat at constant pressure
c_0	constant, Eq. (65)
C	function of β, Eq. (28)
${}_1F_1$	confluent hypergeometric function of the first kind
G	function of ζ and β, Eq. (37)
k	thermal conductivity
L_{th}^*	dimensionless thermal entrance length
Nu	Nusselt number
Pe	Peclet number
q_w	wall heat flux
q_{w0}	uniform wall heat flux
r	radial coordinate
r_0	radius of the duct
s	Laplace transformed axial coordinate
s_n	simple poles of $\tilde{\psi}(\eta, s)$, Eq. (29)
T	temperature
T_e	entrance wall temperature
u_m	average velocity
Y	analytic function defined by Eq. (35)
z	axial coordinate

Greek symbols

α	thermal diffusivity
β	complex variable, Eq. (26)
β_n	positive roots of Eq. (30)
ζ	dimensionless axial coordinate, Eq. (6)
ζ_s	value of ζ corresponding to a singularity of Nu
η	dimensionless radial coordinate, Eq. (6)
θ	dimensionless temperature, Eq. (6)
μ	dynamic viscosity
ν	kinematic viscosity
ξ	dummy integration variable
φ	function of η and β, Eq. (27)
ϕ_w	dimensionless wall heat flux, Eq. (6)
χ	function of η, Eq. (62)
ψ	function of η and ζ, Eq. (20)
Ψ	function of η and ζ, Eq. (11)
Ω_1, Ω_2	functions of ζ, Eqs. (51) and (70)

Superscripts/subscripts

$\sim$	Laplace transform
$'$	derivative of a function with respect to its argument
w	wall value
b	bulk mean value, Eq. (38)

axial station. The reason of these singularities is that, at some specific positions, the wall heat flux may be nonzero while the difference between the wall temperature and the bulk temperature is zero. Stated differently, the traditional definition of the local Nusselt number based on the choice of the bulk temperature as the reference temperature may become pathologic when viscous heating is taken into account [5]. More recently, studies of the viscous dissipation effect in laminar duct flows have been performed in order to include the cases of slug velocity profile, of slip-flow in microtubes and of non-Newtonian fluid behavior [8–11].

The aim of the present paper is to perform an analytical study of the thermal entrance region heat transfer in a circular duct with a prescribed axially-varying wall heat flux. The effect of viscous dissipation is taken into account in a self-consistent way by assuming a non-uniform temperature profile as the initial condition at the entrance axial station. The latter profile is obtained as the fully developed profile determined by an upstream adiabatic preparation of the fluid. It will be shown that this assumption induces strong differences with respect to the classical solutions of the entrance problem with viscous dissipation, which are based on a uniform entrance temperature profile. The solution of the local energy balance equation is obtained analytically by means of the Laplace transform method.

2. Mathematical model

Let us consider laminar Poiseuille flow in a circular duct such that, in the region $z < 0$, the wall is thermally insulated while, in the region $z > 0$, an axisymmetric wall heat flux distribution $q_w(z)$ is prescribed. A sketch of the duct and of the prescribed boundary conditions is given in Fig. 1. Forced convection regime is considered, the effect of axial heat conduction in the fluid and in the wall is neglected, while the effect of viscous dissipation is taken into account.

Under the above assumptions, the governing equations are

$$2\left(1-\frac{r^2}{r_0^2}\right)\frac{\partial T}{\partial z} = \frac{\alpha}{u_m r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{16\nu u_m}{c_p r_0^4} r^2; \tag{1}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \quad \text{(symmetry condition)}; \tag{2}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = 0, \quad z < 0; \tag{3}$$

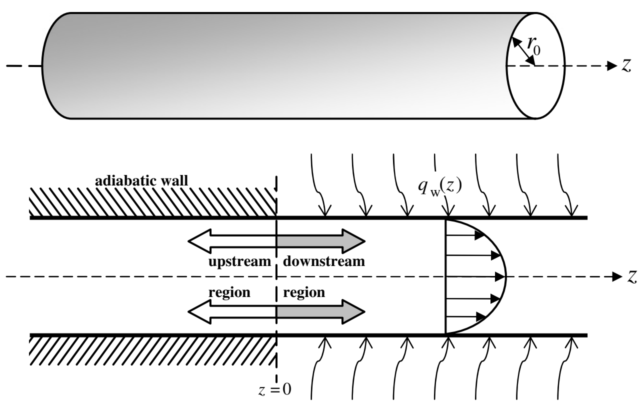

Fig. 1. Drawing of the duct and of the boundary conditions.

$$k\frac{\partial T}{\partial r}\bigg|_{r=r_0} = q_w(z), \quad z > 0. \tag{4}$$

In the region $z < 0$ the hydrodynamic and thermal preparation of the fluid takes place. In the region $z > 0$, a thermal entrance regime determined by the distribution $q_w(z)$ and by the thermal preparation occurs. The region $z < 0$, virtually infinite, is assumed to be sufficiently long for the velocity profile to become axially invariant and for the temperature profile to attain, at least in the vicinity of $z = 0$, the fully developed form compatible with the wall thermal insulation condition. As a consequence of Eqs. (1)–(3), the latter profile is given by

$$T(r,z) = T(r_0,z) - \frac{2\mu u_m^2}{k}\left(1 - \frac{r^2}{r_0^2}\right)^2. \tag{5}$$

Let us introduce the dimensionless quantities

$$\eta = \frac{r}{r_0}, \quad \zeta = \frac{z}{2r_0 Pe}, \quad \theta = \frac{k(T - T_e)}{\mu u_m^2},$$
$$\phi_w = \frac{r_0 q_w}{\mu u_m^2}, \quad Pe = \frac{2r_0 u_m}{\alpha}, \tag{6}$$

where $T_e = T(r_0, 0)$. Then, Eqs. (1), (2), (4) and (5) yield the following parabolic initial value problem in the domain $\zeta > 0$:

$$\frac{1}{2}(1-\eta^2)\frac{\partial\theta}{\partial\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) + 16\eta^2; \tag{7}$$

$$\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0} = 0; \tag{8}$$

$$\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=1} = \phi_w(\zeta); \tag{9}$$

$$\theta(\eta, 0) = -2(1-\eta^2)^2. \tag{10}$$

The solution of Eqs. (7)–(10) can be expressed in the form

$$\theta(\eta,\zeta) = \theta(\eta,0) + 32\zeta + \Psi(\eta,\zeta), \tag{11}$$

where $\Psi(\eta,\zeta)$ is the solution of the initial value problem

$$\frac{1}{2}(1-\eta^2)\frac{\partial\Psi}{\partial\zeta} = \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Psi}{\partial\eta}\right); \tag{12}$$

$$\frac{\partial\Psi}{\partial\eta}\bigg|_{\eta=0} = 0; \tag{13}$$

$$\frac{\partial\Psi}{\partial\eta}\bigg|_{\eta=1} = \phi_w(\zeta); \tag{14}$$

$$\Psi(\eta, 0) = 0. \tag{15}$$

The redefinition (11) of the unknown function in Eqs. (7)–(10) has allowed one to map a non-homogeneous partial differential equation with a non-homogeneous initial condition into a homogeneous partial differential equation with a homogeneous initial condition, namely Eqs. (12) and (15). Thus, one obtains a differential problem, Eqs. (12)–(15), which can be solved in a straightforward way by the Laplace transform method, as it will be shown in the next section. The effectiveness of the redefinition of the unknown variable in Eqs. (7)–(10) is due to the compatibility between the initial condition (10) and the partial differential equation (7). Indeed, the initial temperature profile (10) has been obtained by employing a solution of Eq. (7), i.e. the solution obtained by expressing Eq. (5) in a dimensionless form. It must be pointed out that the efficacy of a redefinition similar to Eq. (11) would have been precluded if one had adopted the assumption of a uniform temperature profile at $z = 0$. In fact, a uniform temperature profile can be in no sense traced back to a solution of Eq. (7). As a consequence, the procedure to solve the present thermal entrance problem by means of the Laplace transform method would be far more complicated in the case of a uniform entrance temperature profile. The extent of this complication can be realized by considering the treatment described in the next section.

3. Analytical solution

Let us define the Laplace transform of function Ψ,

$$\widetilde{\Psi}(\eta,s) = \int_0^{+\infty} \Psi(\eta,\zeta)e^{-s\zeta}d\zeta. \tag{16}$$

On account of the properties of the Laplace transform [12], the governing Eqs. (12)–(15) yield

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\widetilde{\Psi}}{d\eta}\right) - \frac{s}{2}(1-\eta^2)\widetilde{\Psi} = 0; \tag{17}$$

$$\frac{d\widetilde{\Psi}}{d\eta}\bigg|_{\eta=0} = 0; \tag{18}$$

$$\frac{d\widetilde{\Psi}}{d\eta}\bigg|_{\eta=1} = \tilde{\phi}_w(s). \tag{19}$$

3.1. Solution in the Laplace transform domain

The solution of Eq. (17) can be easily expressed in the form

$$\widetilde{\Psi}(\eta,s) = \tilde{\phi}_w(s)\,\tilde{\psi}(\eta,s), \tag{20}$$

where $\tilde{\psi}(\eta, s)$ is the solution of the boundary value problem

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\tilde{\psi}}{d\eta}\right) - \frac{s}{2}(1-\eta^2)\tilde{\psi} = 0; \tag{21}$$

$$\left.\frac{d\tilde{\psi}}{d\eta}\right|_{\eta=0} = 0; \tag{22}$$

$$\left.\frac{d\tilde{\psi}}{d\eta}\right|_{\eta=1} = 1. \tag{23}$$

By invoking the convolution theorem of Laplace transforms [12] and by using Eqs. (10), (11) and (20), the dimensionless temperature $\theta(\eta, \zeta)$ in the thermal entrance region $\zeta > 0$ can be formally expressed as

$$\theta(\eta, \zeta) = -2(1-\eta^2)^2 + 32\zeta + \int_0^\zeta \phi_w(\xi)\,\psi(\eta, \zeta - \xi)\,d\xi. \tag{24}$$

The last task is the solution of the boundary value problem (21)–(23) and the evaluation of function $\psi(\eta, \zeta)$, i.e. of the inverse Laplace transform of $\tilde{\psi}(\eta, s)$.

The general solution of Eq. (21) that fulfils Eq. (22) can be expressed in terms of the confluent hypergeometric function of the first kind

$$\tilde{\psi}(\eta, s) = \frac{\varphi(\eta, \beta)}{C(\beta)}, \tag{25}$$

$$\beta^2 = -2s, \tag{26}$$

$$\varphi(\eta, \beta) = e^{-\beta\eta^2/4}\, {}_1F_1\left(\frac{4-\beta}{8}, 1; \frac{\beta\eta^2}{2}\right), \tag{27}$$

where $C(\beta)$ is an integration constant that can be determined by applying the boundary condition (23). In fact, by employing the properties of the confluent hypergeometric function [13], one obtains

$$C(\beta) = \left.\frac{d\varphi}{d\eta}\right|_{\eta=1} = -\frac{\beta}{8}e^{-\beta/4}\left[4\,{}_1F_1\left(\frac{4-\beta}{8}, 1; \frac{\beta}{2}\right) + (\beta-4)\,{}_1F_1\left(\frac{12-\beta}{8}, 2; \frac{\beta}{2}\right)\right]. \tag{28}$$

3.2. Inversion of the Laplace transform

On account of Eqs. (25)–(28), it can be shown that $\tilde{\psi}(\eta, s)$ is a function of the complex variable s which admits an infinite sequence of real simple poles for

$$s = s_n = -\frac{\beta_n^2}{2}, \quad n = 0, 1, 2, \ldots, \tag{29}$$

where β_0 is 0, while, for $n > 0$, β_n are the positive roots of the equation

$$C(\beta) = 0. \tag{30}$$

The first 20 positive roots β_n are reported in Table 1. By employing the usual method for the inversion of Laplace transforms based on the integration along the Bromwich contour [12], one can express the inverse Laplace transform of $\tilde{\psi}(\eta, s)$ as the sum of the residues of $e^{s\zeta}\tilde{\psi}(\eta, s)$ evaluated for all the poles $s = s_n$, namely

Table 1
The first 20 values of β_n and $C'(\beta_n)$

n	β_n	$C'(\beta_n)$
1	10.1350110019	0.489080441271
2	18.3152128526	−0.623601390240
3	26.3944494701	0.717608404264
4	34.4404587279	−0.792430816789
5	42.4710345631	0.855630572841
6	50.4930623658	−0.910887946159
7	58.5098111032	0.960308860407
8	66.5230474681	−1.00522572585
9	74.5338164220	1.04654178666
10	82.5427786917	−1.08490110043
11	90.5503741204	1.12078103927
12	98.5569077531	−1.15454643994
13	106.562598334	1.18648315907
14	114.567607139	−1.21681983842
15	122.572055898	1.24574256300
16	130.576038355	−1.27340504808
17	138.579627971	1.29993591089
18	146.582883195	−1.32544398203
19	154.585851159	1.35002226377
20	162.588570327	−1.37375093221

$$\psi(\eta, \zeta) = \sum_{n=0}^{\infty} \mathrm{Res}\left[e^{s\zeta}\tilde{\psi}(\eta, s); s = -\frac{\beta_n^2}{2}\right]. \tag{31}$$

On account of Eqs. (25)–(28), the residues are easily determined

$$\mathrm{Res}\left[e^{s\zeta}\tilde{\psi}(\eta, s); s = -\frac{\beta_0^2}{2} = 0\right] = 8, \tag{32}$$

$$\mathrm{Res}\left[e^{s\zeta}\tilde{\psi}(\eta, s); s = -\frac{\beta_n^2}{2}\right] = -\frac{\beta_n e^{-\beta_n^2\zeta/2}\varphi(\eta, \beta_n)}{C'(\beta_n)}, \quad n = 1, 2, 3, \ldots \tag{33}$$

The derivative $C'(\beta)$ can be evaluated on account of Eq. (28) and of the properties of the confluent hypergeometric function [13],

$$C'(\beta) = \frac{1}{256}e^{-\beta/4}\left\{16(8 - 8\beta + \beta^2)\,{}_1F_1\left(\frac{12-\beta}{8}, 2; \frac{\beta}{2}\right) + (\beta-4)\left[(\beta-12)\beta\,{}_1F_1\left(\frac{20-\beta}{8}, 3; \frac{\beta}{2}\right) + 32\,{}_1F_1\left(\frac{4-\beta}{8}, 1; \frac{\beta}{2}\right) + 16\beta Y\left(\frac{4-\beta}{8}, 1; \frac{\beta}{2}\right) + 4(\beta-4)\beta Y\left(\frac{12-\beta}{8}, 2; \frac{\beta}{2}\right)\right]\right\}, \tag{34}$$

where $Y(a, b; x)$ is the analytic function

$$Y(a, b; x) = \frac{\partial}{\partial a}\,{}_1F_1(a, b; x). \tag{35}$$

The first 20 values of $C'(\beta_n)$ are given in Table 1. By employing Eqs. (31)–(33), Eq. (24) can be rewritten as

$$\theta(\eta,\zeta) = -2(1-\eta^2)^2 + 32\zeta + 8\int_0^\zeta \phi_w(\xi)\,d\xi - \sum_{n=1}^{\infty}\frac{\beta_n}{C'(\beta_n)}\varphi(\eta,\beta_n)\,G(\zeta,\beta_n)\,e^{-\beta_n^2\zeta/2}, \tag{36}$$

where function $G(\zeta,\beta)$ is given by

$$G(\zeta,\beta) = \int_0^\zeta \phi_w(\xi)\,e^{\beta^2\xi/2}\,d\xi. \tag{37}$$

3.3. Nusselt number

Let us introduce the bulk temperature,

$$T_b = 4\int_0^1 \eta(1-\eta^2)T\,d\eta. \tag{38}$$

Then, on account of Eqs. (21), (25) and (30), one can determine the dimensionless difference between the wall temperature and the bulk temperature,

$$\theta_w - \theta_b = \frac{k(T_w - T_b)}{\mu u_m^2} = 1 - \sum_{n=1}^{\infty}\frac{\beta_n}{C'(\beta_n)}\varphi(1,\beta_n)\,G(\zeta,\beta_n)\,e^{-\beta_n^2\zeta/2}. \tag{39}$$

As a consequence, the Nusselt number is given by

$$Nu = \frac{2r_0 q_w}{k(T_w - T_b)} = \frac{2\phi_w}{\theta_w - \theta_b} = 2\phi_w(\zeta)\left[1 - \sum_{n=1}^{\infty}\frac{\beta_n}{C'(\beta_n)}\varphi(1,\beta_n)\,G(\zeta,\beta_n)\,e^{-\beta_n^2\zeta/2}\right]^{-1}. \tag{40}$$

The expressions of the dimensionless temperature field and of the local Nusselt number given by Eqs. (36) and (40) refer to a general axial distribution of the wall heat flux $\phi_w(\zeta)$. In the following, two special cases are investigated in detail: a uniform wall heat flux, a linearly varying wall heat flux.

4. Uniform wall heat flux

If the wall heat flux distribution is a constant, $q_w(z) = q_{w0}$, one can define the Brinkman number such that

$$\phi_w(\zeta) = \frac{r_0 q_{w0}}{\mu u_m^2} = \frac{1}{Br}. \tag{41}$$

As a consequence of Eqs. (37) and (41), the function $G(\zeta,\beta)$ can be expressed as

$$G(\zeta,\beta) = \frac{2}{Br\beta^2}\left(e^{\beta^2\zeta/2} - 1\right). \tag{42}$$

Therefore, Eqs. (36) and (39) can be rewritten as

$$\theta(\eta,\zeta) = -2(1-\eta^2)^2 + 8\left(4+\frac{1}{Br}\right)\zeta - \frac{2}{Br}\sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n C'(\beta_n)} + \frac{2}{Br}\sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n C'(\beta_n)}e^{-\beta_n^2\zeta/2}, \tag{43}$$

$$\theta_w - \theta_b = 1 - \frac{2}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n C'(\beta_n)} + \frac{2}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n C'(\beta_n)}e^{-\beta_n^2\zeta/2}. \tag{44}$$

Far away from the entrance cross-section $\zeta = 0$, strictly speaking in the limit $\zeta \to +\infty$, the axial gradient of temperature becomes a constant, $\partial\theta/\partial\zeta = \text{constant}$, so that the dimensionless temperature field is uniquely determined by Eqs. (7)–(9) and is such that

$$\lim_{\zeta\to+\infty}[\theta(1,\zeta) - \theta(\eta,\zeta)] = 2(1-\eta^2)^2 + \frac{1}{4Br}(1-\eta^2)(3-\eta^2), \tag{45}$$

$$\lim_{\zeta\to+\infty}[\theta_w - \theta_b] = 1 + \frac{11}{24Br}. \tag{46}$$

A comparison between Eqs. (43)–(46) leads to the following sum rule:

$$\sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n C'(\beta_n)} = -\frac{11}{48} + \frac{1}{8}(1-\eta^2)(3-\eta^2). \tag{47}$$

This sum rule is very useful since the series on the left hand side of Eq. (47) has a very poor convergence especially for $\eta = 1$ where truncation to the first 100 terms yields a result with a relative error of 3.6%. On account of Eq. (47), Eqs. (43) and (44) yield

$$\theta(\eta,\zeta) = -2(1-\eta^2)^2 + \frac{1}{4Br}\left[\frac{11}{6} - (1-\eta^2)(3-\eta^2)\right] + 8\left(4+\frac{1}{Br}\right)\zeta + \frac{2}{Br}\sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n C'(\beta_n)}e^{-\beta_n^2\zeta/2}, \tag{48}$$

$$\theta_w - \theta_b = 1 + \frac{11}{24Br} + \frac{2}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n C'(\beta_n)}e^{-\beta_n^2\zeta/2}. \tag{49}$$

By employing Eqs. (40) and (49), the local Nusselt number is expressed as

$$Nu(\zeta) = \frac{48}{24Br + 11 - 48\Omega_1(\zeta)}, \tag{50}$$

where

$$\Omega_1(\zeta) = -\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n C'(\beta_n)}e^{-\beta_n^2\zeta/2}. \tag{51}$$

In the case $Br = 0$, the values of $Nu(\zeta)$ determined by means of Eqs. (50) and (51) are in perfect agreement with those reported in Ref. [14] with reference to the hypothesis of negligible viscous dissipation (see Table 2). Moreover, in

Table 2
Values of $Nu(\zeta)$ for uniform wall heat flux and $Br = 0$

ζ	$Nu(\zeta)$	
	Present paper	Ref. [14]
0.00005	34.51065	34.511
0.0001	27.27564	27.276
0.0005	15.81273	15.813
0.001	12.53816	12.538
0.005	7.493677	7.4937
0.01	6.148144	6.1481
0.02	5.198390	5.1984
0.03	4.815668	4.8157
0.04	4.621309	4.6213
0.05	4.513886	4.5139
0.1	4.374793	4.3748
0.2	4.363702	4.3637
0.5	4.363636	–

this case, the dimensionless thermal entrance length, i.e. the value of ζ such that Nu is 0.05% greater than the asymptotic value 48/11, can be easily evaluated as

$$L^*_{\rm th} = 0.043052765, \tag{52}$$

in perfect agreement with the value reported in Ref. [7].

On account of Eq. (47), one can conclude that function $\Omega_1(\zeta)$ has the following properties:

$$\Omega_1(0) = \frac{11}{48}, \quad \lim_{\zeta\to+\infty} \Omega_1(\zeta) = 0. \tag{53}$$

As it is shown in Table 3, $\Omega_1(\zeta)$ is a monotonic decreasing function of ζ. These simple features of $\Omega_1(\zeta)$ allow one to predict the qualitative behavior of the local Nusselt number both for positive and for negative values of the Brinkman number. In general, Eqs. (50) and (53) lead to the conclusion that

$$Nu(0) = \frac{2}{Br}, \quad \lim_{\zeta\to+\infty} Nu(\zeta) = \frac{48}{24Br + 11}. \tag{54}$$

Table 3
Values of $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$

ζ	$\Omega_1(\zeta)$	$\Omega_2(\zeta) \times 10^2$
0	0.229167	0.111762
0.00005	0.200190	0.111243
0.0001	0.192504	0.110753
0.0005	0.165926	0.107214
0.001	0.149410	0.103289
0.005	0.0957208	0.0798017
0.01	0.0665160	0.0598910
0.02	0.0367994	0.0349942
0.03	0.0215112	0.0207888
0.04	0.0127777	0.0124110
0.05	0.00762807	0.00742086
0.06	0.00456095	0.00443925
0.07	0.00272840	0.00265601
0.08	0.00163240	0.00158917
0.09	0.000976716	0.000950860
0.1	0.000584406	0.000568939
0.2	3.43725×10^{-6}	3.34628×10^{-6}
0.5	6.99363×10^{-13}	6.80854×10^{-13}
∞	0	0

4.1. Case $Br > 0$

For positive values of Br, Eqs. (50) and (53) allow one to infer that Nu is a monotonically decreasing function of ζ, free of singularities. Plots of Nu versus ζ for different positive values of Br are sketched in Fig. 2. This figure shows that, by increasing the value of Br, the local Nusselt number distribution in the entrance region tend to become more and more uniform. This feature is consistent with the significance of an increasing value of Br: the larger is the Brinkman number the smaller is the uniform wall heat flux prescribed. Hence, a large Brinkman number corresponds to a slight change of the boundary condition from the upstream region (adiabatic wall) to the downstream region (nonzero wall heat flux).

4.2. Case $-11/24 < Br < 0$

In this case, on account of Eqs. (50) and (53), the following behavior of Nu is predicted. For small values of ζ, $Nu(\zeta)$ is negative and monotonically decreasing. The local Nusselt number has a vertical asymptote (singularity) for an axial position $\zeta = \zeta_s$ which depends on Br and is such that

$$\Omega_1(\zeta_s) = \frac{Br}{2} + \frac{11}{48}. \tag{55}$$

For $\zeta > \zeta_s$, $Nu(\zeta)$ is positive and decreases monotonically with ζ. Special cases are given by the limit $Br \to 0$, when the axial position ζ_s tends to 0, and by the limit $Br \to -11/24$, when the axial position ζ_s tends to $+\infty$. These features are consistent with Eq. (54). The physical reason of the singularity of Nu is the existence of an axial position where the difference $\theta_w - \theta_b$ vanishes. Such a circumstance, is expected to occur for sufficiently high values of the subtracted wall heat flux (small negative values of Br), because the asymptotic value of θ_b is higher than that of θ_w, due to the internal heat generation. Since, at $\zeta = 0$, the reverse

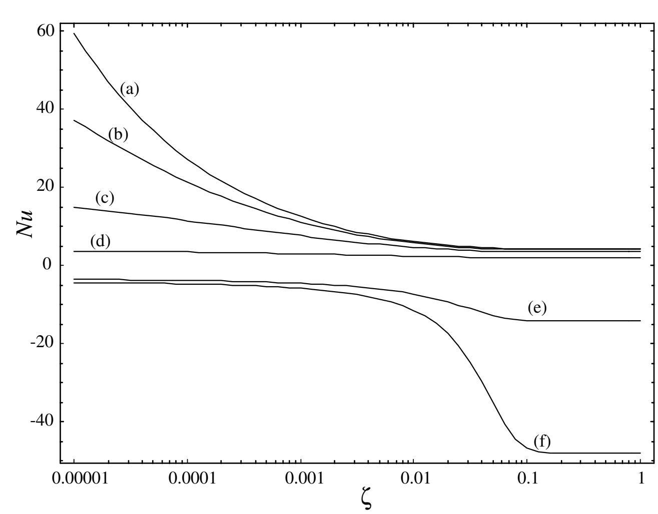

Fig. 2. Uniform wall heat flux: plots of Nu versus ζ for $Br = 0$ (a), $Br = 0.02$ (b), $Br = 0.1$ (c), $Br = 0.5$ (d), $Br = -0.6$ (e), $Br = -0.5$ (f).

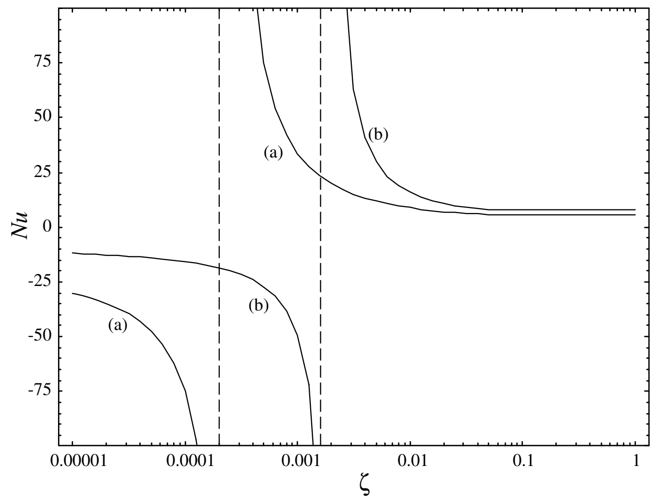

Fig. 3. Uniform wall heat flux: plots of Nu versus ζ for $Br = -0.1$ (a), $Br = -0.2$ (b).

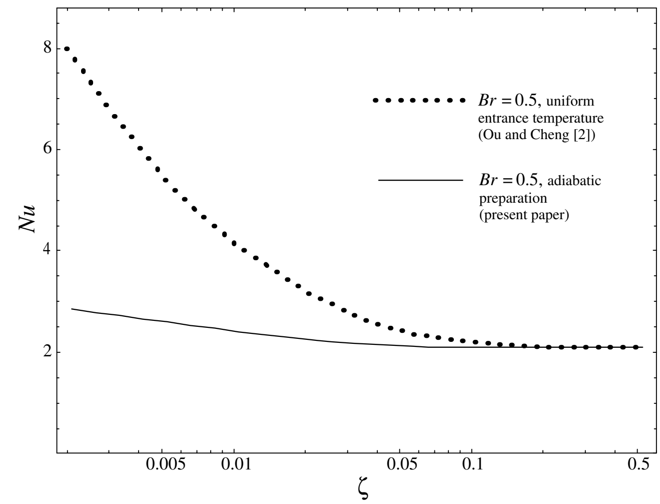

Fig. 4. Uniform wall heat flux: plots of Nu versus ζ for $Br = 0.5$. Comparison between the uniform entrance temperature case (Ou and Cheng [2]) and the adiabatic preparation case (present paper).

occurs ($\theta_b < \theta_w$), an axial station where the quantities θ_b and θ_w are equal must exist. Fig. 3 refers to the range $-11/24 < Br < 0$ and displays the behavior of $Nu(\zeta)$ in two cases: $Br = -0.1$, $Br = -0.2$.

4.3. Case $Br < -11/24$

As a consequence of Eqs. (50) and (53), in this case, Nu is a negative monotonically decreasing function of ζ, free of singularities. Plots of Nu versus ζ for different negative values of Br smaller than $-11/24 \cong -0.458333$ are sketched in Fig. 2. As in the case $Br > 0$, also for $Br < -11/24$ one concludes that an increase of $|Br|$ yields a more uniform distribution of the local Nusselt number.

4.4. Sensitivity to the initial condition

The results obtained are based on the assumption of a non-uniform initial temperature distribution in the entrance section. In the literature, other analytical solutions for uniform wall heat flux have been obtained by using separation of variables, in the case of a uniform entrance temperature distribution [2,6]. The sensitivity of the solution to the initial condition can be estimated by comparing the results obtained in the present section with those provided, for instance, by Ou and Cheng [2]. The asymptotic value of Nu reached for $\zeta \to +\infty$ does not depend on the initial condition and, in fact, is the same in the present paper and in Ref. [2]. For negative values of Br, strong differences between the present results and those reported by Ou and Cheng [2] exist. Indeed, these authors show that, for $Br < -11/24$, singularities of the local Nusselt number arise at some axial station. These singularities are due to the vanishing of the difference $T_w - T_b$ at an axial distance from the entrance cross-section that depends on the value of Br. On the other hand, no singularities of Nu arise in the present case of adiabatic preparation for $Br < -11/24$, as specified in Section 4.3. For positive values of the Brinkman number, the dependence on the initial condition is also very strong as is shown in Fig. 4, where the Ou and Cheng evaluation of $Nu(\zeta)$ for $Br = 0.5$ is compared with the one obtained in the present paper.

5. Linearly varying wall heat flux

If the wall heat flux is a linear function of z, $q_w(z) = q_{w0}\zeta$, function $\phi_w(\zeta)$ can be expressed as

$$\phi_w(\zeta) = \frac{r_0 q_{w0}}{\mu u_m^2}\zeta = \frac{\zeta}{Br}. \tag{56}$$

As a consequence of Eqs. (37) and (56), the function $G(\zeta, \beta)$ can be expressed as

$$G(\zeta, \beta) = \frac{2}{Br\beta^4}\left[2 + (\beta^2\zeta - 2)e^{\beta^2\zeta/2}\right]. \tag{57}$$

Therefore, Eqs. (36) and (39) can be rewritten as

$$\begin{aligned}\theta(\eta, \zeta) = {} & -2(1-\eta^2)^2 + \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(\eta, \beta_n)}{\beta_n^3 C'(\beta_n)} \\ & + \left[32 + \frac{11 - 6(1-\eta^2)(3-\eta^2)}{24Br}\right]\zeta + \frac{4}{Br}\zeta^2 \\ & - \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(\eta, \beta_n)}{\beta_n^3 C'(\beta_n)}e^{-\beta_n^2\zeta/2},\end{aligned} \tag{58}$$

$$\begin{aligned}\theta_w - \theta_b = {} & 1 + \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1, \beta_n)}{\beta_n^3 C'(\beta_n)} + \frac{11}{24Br}\zeta \\ & - \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1, \beta_n)}{\beta_n^3 C'(\beta_n)}e^{-\beta_n^2\zeta/2}.\end{aligned} \tag{59}$$

Eqs. (58) and (59) allow one to infer that, for large ζ, i.e. in the fully-developed regime, the following relations hold:

$$\theta(1,\zeta) - \theta(\eta,\zeta) \cong \chi(\eta) + \frac{(1-\eta^2)(3-\eta^2)}{4Br}\zeta, \tag{60}$$

$$\theta_w - \theta_b \cong c_0 + \frac{11}{24Br}\zeta, \tag{61}$$

where $\chi(\eta)$ and c_0 are given by

$$\chi(\eta) = 2\left(1-\eta^2\right)^2 + \frac{4}{Br}\left[\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n^3 C'(\beta_n)} - \sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n^3 C'(\beta_n)}\right], \tag{62}$$

$$c_0 = 1 + \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n^3 C'(\beta_n)}. \tag{63}$$

On the other hand, by imposing compatibility between Eq. (60) and the governing Eqs. (7)–(9), one obtains

$$\chi(\eta) = 2(1-\eta^2)^2 - \frac{(1-\eta^2)^3(53-9\eta^2)}{4608Br}, \tag{64}$$

$$c_0 = 1 - \frac{103}{23,040Br}. \tag{65}$$

A comparison between Eqs. (62)–(65) allows one to obtain the following sum rule:

$$\sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n^3 C'(\beta_n)} = -\frac{103}{92,160} + \frac{(1-\eta^2)^3(53-9\eta^2)}{18,432}. \tag{66}$$

By employing this sum rule, Eqs. (58) and (59) can be rewritten as

$$\theta(\eta,\zeta) = -\frac{103}{23,040Br} - 2(1-\eta^2)^2 + \frac{(1-\eta^2)^3(53-9\eta^2)}{4608Br} + \left[32 + \frac{11-6(1-\eta^2)(3-\eta^2)}{24Br}\right]\zeta + \frac{4}{Br}\zeta^2 - \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(\eta,\beta_n)}{\beta_n^3 C'(\beta_n)}e^{-\beta_n^2\zeta/2}, \tag{67}$$

$$\theta_w - \theta_b = 1 - \frac{103}{23,040Br} + \frac{11}{24Br}\zeta - \frac{4}{Br}\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n^3 C'(\beta_n)}e^{-\beta_n^2\zeta/2}. \tag{68}$$

As a consequence of Eqs. (40), (56) and (68), the local Nusselt number can be expressed as

$$Nu(\zeta) = \frac{46,080\zeta}{23,040Br - 103 + 10,560\zeta + 92,160\Omega_2(\zeta)}, \tag{69}$$

where

$$\Omega_2(\zeta) = -\sum_{n=1}^{\infty}\frac{\varphi(1,\beta_n)}{\beta_n^3 C'(\beta_n)}e^{-\beta_n^2\zeta/2}. \tag{70}$$

As a consequence of Eqs. (51) and (70), it can be easily shown that

$$\Omega_2'(\zeta) = -\frac{1}{2}\Omega_1(\zeta). \tag{71}$$

On account of Eq. (66), one can conclude that function $\Omega_2(\zeta)$ has the following properties:

$$\Omega_2(0) = \frac{103}{92,160}, \quad \lim_{\zeta\to+\infty}\Omega_2(\zeta) = 0. \tag{72}$$

As it is shown in Table 3, $\Omega_2(\zeta)$ is a monotonic decreasing function of ζ. These simple features of $\Omega_2(\zeta)$ allow one to predict the qualitative behavior of the local Nusselt number both for positive and for negative values of the Brinkman number. In general, Eqs. (69), (71) and (72) lead to the conclusion that

$$\lim_{\zeta\to+\infty} Nu(\zeta) = \frac{48}{11}, \tag{73}$$

whatever is the value of the Brinkman number, while

$$Nu(0) = \begin{cases} 0, & Br \neq 0 \\ +\infty, & Br = 0. \end{cases} \tag{74}$$

The existence of the fully developed value of the Nusselt number given by Eq. (73) was predicted by using asymptotic methods in Ref. [15].

5.1. Case $Br > 0$

For positive values of Br, function $Nu(\zeta)$ is free of singularities. If

$$Br > \frac{103}{23,040} \cong 0.00447049, \tag{75}$$

then Eq. (69) allows one to infer that Nu is a monotonically increasing function of ζ. On the other hand, if

$$0 < Br < \frac{103}{23,040}, \tag{76}$$

the local Nusselt number initially undergoes a rapid increase with ζ, reaches a maximum, and then decreases attaining asymptotically the fully developed value 48/11. The smaller is the value of Br, the smaller is the value of ζ where Nu is maximum. In the limit $Br \to 0$, the maximum degenerates into a singularity in the entrance cross-section $\zeta = 0$, coherently with Eq. (74). In the latter limit, which corresponds to a negligible effect of viscous dissipation, the dimensionless thermal entrance length is given by

$$L^*_{th} = 0.20482476. \tag{77}$$

Then, in the case of linearly varying wall heat flux, the value of L^*_{th} is considerably greater than in the case of uniform wall heat flux (see Eq. (52)).

Plots of $Nu(\zeta)$ are reported in Fig. 5 with reference to different positive values of Br. These plots show how the thermal entrance region becomes more and more expanded as the value of Br increases. This feature is completely expected since, as Br increases, the wall heat flux becomes a slower and slower increasing function of ζ. As a consequence, if Br has a high value, it takes a very long axial distance for the wall heat flux to attain a value comparable to the viscous dissipation heating contribution. The special case $Br \to 0$ is represented in Fig. 6.

5.2. Case $Br < 0$

For negative values of Br, the local Nusselt number becomes singular at some axial station. The singularity

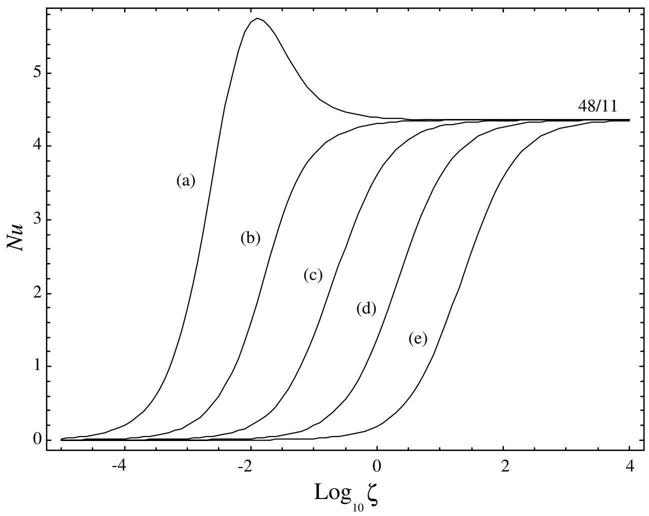

Fig. 5. Linearly varying wall heat flux: plots of Nu versus ζ for $Br = 0.001$ (a), $Br = 0.01$ (b), $Br = 0.1$ (c), $Br = 1$ (d), $Br = 10$ (e).

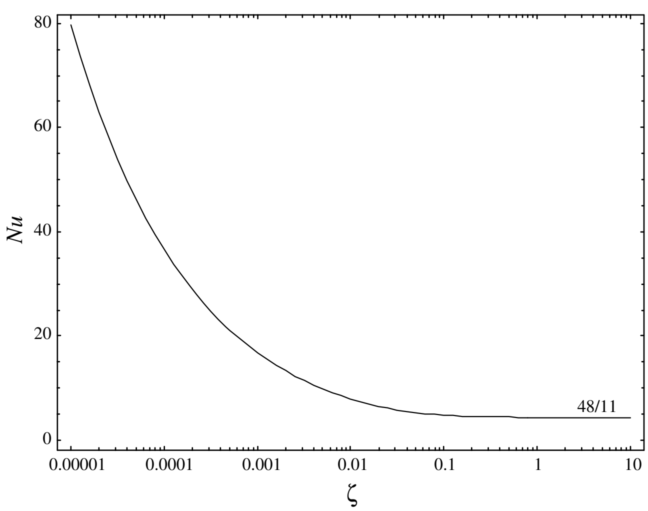

Fig. 6. Linearly varying wall heat flux: plot of Nu versus ζ for $Br = 0$.

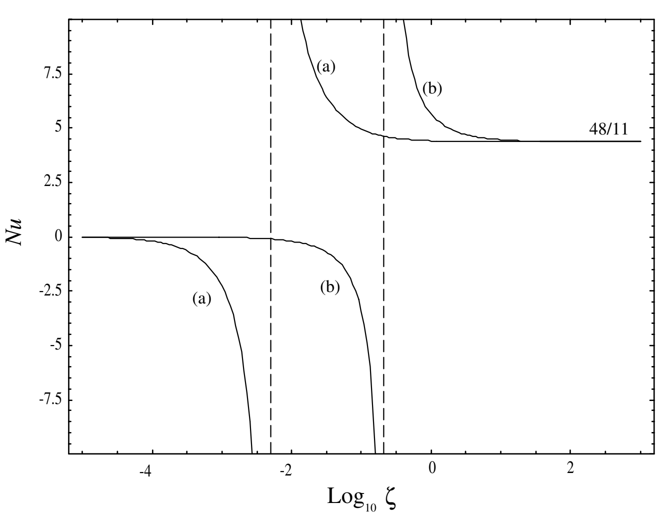

Fig. 7. Linearly varying wall heat flux: plots of Nu versus ζ for $Br = -0.001$ (a), $Br = -0.1$ (b).

arises where the denominator of the fractional expression of the Nusselt number Eq. (69) becomes zero. For sufficiently high values of $|Br|$ ($|Br| \gtrsim 0.05$), the axial position of this singularity, $\zeta = \zeta_s$, can be evaluated with a fair approximation as

$$\zeta_s \cong \frac{103 + 23,040|Br|}{10,560}. \tag{78}$$

Examples of the singular behavior of the local Nusselt number for $Br < 0$ are sketched in Fig. 7, where the two cases $Br = -0.001$ and $Br = -0.1$ are considered. This figure shows that the same expansion of the thermal entrance region when $|Br|$ increases, observed for positive values of Br, occurs also if $Br < 0$.

6. Considerations on the Brinkman number

The Brinkman number, defined in Eq. (41), is a parameter inversely proportional to the reference wall heat flux q_{w0}, that characterizes the thermal boundary conditions at the wall. Hence, in principle, any arbitrarily high value of Br can be found in an actual flow provided that the reference wall heat flux q_{w0} is sufficiently small. However, the possibility to obtain very high values of Br may conflict with the practical difficulties in producing conditions of extremely low wall heat flux. This is the reason why, in the present study, cases such that $Br \leqslant 10$ have been considered. Indeed, the value $Br = 10$ is very high, but it can be reasonable in some cases, as it will be shown in the following two examples.

6.1. First example

Let us consider the flow of a highly viscous fluid, castor oil, in a duct with $r_0 = 1$ mm. If the fluid temperature is approximately 20 °C, the fluid viscosity is $\mu = 0.986$ Pa s. By assuming $u_m = 0.1\ \mathrm{m\,s^{-1}}$, one obtains that $Br = 10$ would mean $q_{w0} = 0.986\ \mathrm{W\,m^{-2}}$. Although small, this value of wall heat flux appears to be feasible in an experimental apparatus.

6.2. Second example

Let us consider the flow of water in a microchannel, namely in a duct with $r_0 = 10^{-6}$ m. If the water temperature is approximately 20 °C, its viscosity is $\mu = 1.002 \times 10^{-3}$ Pa s. By assuming $u_m = 0.1\ \mathrm{m\,s^{-1}}$, one obtains that $Br = 10$ would mean $q_{w0} = 1.002\ \mathrm{W\,m^{-2}}$. This value of wall heat flux is approximately equal to that obtained in the first example. However, it appears that this thermal condition is even easier to be obtained in practice due to the rather small diameter of the duct.

7. Conclusions

The laminar forced convection in the thermal entrance region of a circular duct has been investigated by taking into account the effect of viscous dissipation in a self-consistent way. More precisely, the initial condition assumed at the thermal entrance section has been determined by assuming an adiabatic preparation of the fluid in the upstream region. This approach, which differs from traditional treatments of the same problem, ensures that the initial condition can be, at least in principle, reproduced in a possible experimental validation of the results. A similar possibility does not exist whenever the same problem is solved assuming a uniform initial temperature in the entrance section. In fact, a fluid with viscous heating in laminar hydrodynamically developed flow cannot be prepared with a uniform temperature profile.

The local energy balance equation has been solved in the downstream region, by considering a general axially-varying wall heat flux distribution. The solution has been obtained analytically by the Laplace transform method. The general solution has been applied to a pair of special cases: uniform wall heat flux, linearly varying wall heat flux.

In the case of uniform wall heat flux, it has been pointed out that

- the distribution of the local Nusselt number is rather sensitive to the initial condition, as it becomes apparent by comparing the results obtained with those available in the literature for the case of uniform entrance temperature;
- the distribution of the local Nusselt number becomes more and more uniform as the value of the Brinkman number Br increases.

On the other hand, in the case of linearly varying wall heat flux, it has been shown that

- the local Nusselt number attains asymptotically the fully developed value $48/11$, whatever is the value of Br;
- the thermal entrance region tends to increase its length as the value of $|Br|$ increases.

References

[1] H.C. Brinkman, Heat effects in capillary flow I, Appl. Sci. Res. A 2 (1951) 120–124.
[2] J.W. Ou, K.C. Cheng, Viscous dissipation effects on thermal entrance region heat transfer in pipes with uniform wall heat flux, Appl. Sci. Res. 28 (1973) 289–301.
[3] J.W. Ou, K.C. Cheng, Effects of pressure work and viscous dissipation on Graetz problem for gas flows in parallel-plate channels, Wärme Stoffübertrag 6 (1973) 191–198.
[4] J.W. Ou, K.C. Cheng, Viscous dissipation effects on thermal entrance heat transfer in laminar and turbulent pipe flows with uniform wall temperature, in: AIAA/ASME 1974 Thermophysics and Heat Transfer Conference, AIAA Paper No. 74-743, ASME Paper No. 74-HT-50, 1974.
[5] T.F. Lin, K.H. Hawks, W. Leidenfrost, Analysis of viscous dissipation effect on thermal entrance heat transfer in laminar pipe flows with convective boundary conditions, Wärme Stoffübertrag. 17 (1983) 97–105.
[6] T. Basu, D.N. Roy, Laminar heat transfer in a tube with viscous dissipation, Int. J. Heat Mass Transfer 28 (1985) 699–701.
[7] R.K. Shah, A.L. London, Laminar flow forced convection in ducts, Advances in Heat Transfer, Academic Press, New York, 1978.
[8] A. Barletta, E. Zanchini, Forced convection in the thermal entrance region of a circular duct with slug flow and viscous dissipation, Int. J. Heat Mass Transfer 40 (1997) 1181–1190.
[9] A. Barletta, Slug flow heat transfer in circular ducts with viscous dissipation and convective boundary conditions, Int. J. Heat Mass Transfer 40 (1997) 4219–4228.
[10] G. Tunc, Y. Bayazitoglu, Heat transfer in microtubes with viscous dissipation, Int. J. Heat Mass Transfer 44 (2001) 2395–2403.
[11] P.J. Oliveira, P.M. Coelho, F.T. Pinho, The Graetz problem with viscous dissipation for FENE-P fluids, J. Non-Newtonian Fluid Mech. 121 (2004) 69–72.
[12] L. Debnath, Integral Transforms and Their Applications, CRC Press, 1995.
[13] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, DC, 1972.
[14] R.M. Cotta, M.N. Özişik, Laminar forced convection to non-Newtonian fluids in ducts with prescribed wall heat flux, Int. Commun. Heat Mass Transfer 13 (1986) 325–334.
[15] A. Barletta, Fully developed laminar forced convection in circular ducts for power-law fluids with viscous dissipation, Int. J. Heat Mass Transfer 40 (1997) 15–26.